

Differential Geometry

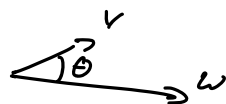
Vector spaces: Inner product $\leadsto \langle \cdot, \cdot \rangle$

Norm:

$$\| \cdot \|^2 = \langle \cdot, \cdot \rangle$$

Angle:

$$\cos \theta = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$$



$T_x X = \{ \text{tangent vectors at } x \}$

Riemannian metric: $\langle \cdot, \cdot \rangle_x$ at every $T_x X$

Riemannian metric

$\langle \cdot, \cdot \rangle_x$ on $T_x X$

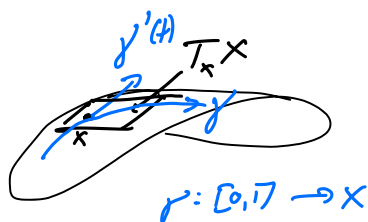
$\leadsto \| \cdot \|$ on $T_x X$

$\leadsto$ Angles

Angles:

$\leadsto$ Length of curve:

$$l(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)} dt$$



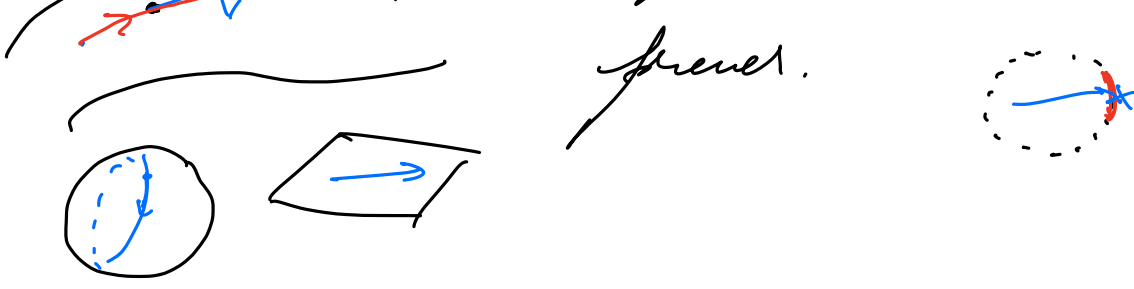
$\leadsto$ Metric space

$$d(p, q) := \inf \{ l(\gamma) \mid \gamma(0) = p, \gamma(1) = q \}$$

Goal: Upgrade from R.S. to R.S. + Riemannian metric
(complete
(conformal metric))

Geodesic: Every tangent vector defines a trajectory in that direction that is locally shortest paths

Completion: Geodesics go on



Prop. X has a metric g , $\Gamma \subset \text{Isom}^+(X, g)$, Γ acts freely + prop. discontin. $\Rightarrow X/\Gamma$ inherits the metric g

↳ ex: $\mathbb{H}^2$ by isometries preserves geodesics

$\mathbb{C} \xrightarrow{z \mapsto z+1} \mathbb{C}/\mathbb{Z}$

if we want to measure angle, we apply τ^{-1} ; and different choices only differ via isometry.

Ex: $\hat{\mathbb{C}}$, sph. metric

Ex: $\mathbb{C}$, Euclidean metric, $\mathbb{C}/\Lambda$, $\Lambda = \{z \mapsto z+b\}$

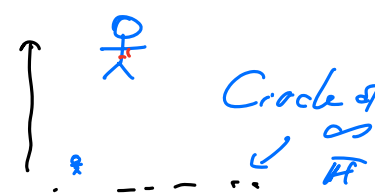
$\mathbb{C}/\Lambda$ gets Euclidean geo

Hyperbolic metric

Al: $g(v, w) = \frac{v \cdot w}{y^2}$, $|v|_{\text{hyp}} = \frac{|v|}{y}$ $\mathbb{H}$

$y > 0$

Complete, conf., curvature -1



Claim. $\text{Isom}^+ \mathbb{H} = \text{PSL}(2, \mathbb{R}) = \text{Aut}(\mathbb{H})$ Möbius are isometries

• We have $\text{Isom}^+ \subset \text{Aut}$

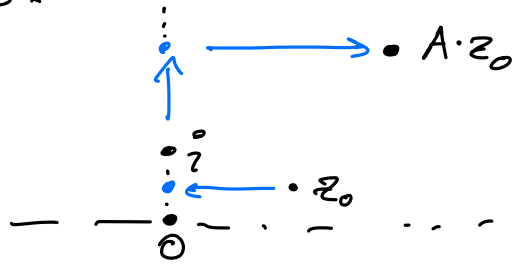
1. $z \mapsto z+r$, $r \in \mathbb{R}$ is an isometry

$z \mapsto z+r$

2. $z \mapsto Kz$, $K \in \mathbb{R}$ is called scaling:

$$g_{Kz}(k \cdot v, k \cdot w) = \frac{(Kv) \cdot (Kw)}{\text{Im}(Kz)^2} = \frac{v \cdot w}{\text{Im}(z)^2} = g_z(v, w)$$

3. $A \in \text{PSL}(2, \mathbb{R}) \iff B \in \text{Isom}^+ \mathbb{H}$



$$B \cdot z_0 = A \cdot z_0$$

$$\Rightarrow B^{-1}A \in \text{PSL}_2 \mathbb{R} \text{ fixes } z_0$$

$$\mathbb{H} \xrightarrow{B^{-1}A} \mathbb{H}$$

$$\cong \uparrow \quad \downarrow \cong$$

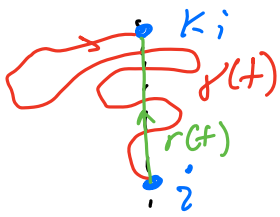
$$\mathbb{D} \xrightarrow{\sim} \mathbb{D} \text{ fixes a point.}$$

Schwarz lemma $\Rightarrow$ Rotator $\Rightarrow |(B^{-1}A)'(z_0)| = 1$, $B^{-1}A$ is an isometry at z_0 , so A is at z_0 .

Cor. Hyp. R.S.'s have a hyp. geometry!

pf. $X \cong \mathbb{H}/\Gamma$, $\Gamma \subset \text{Aut } \mathbb{H} = \text{Isom}^+ \mathbb{H}$, so X inherits the hyp. geo.

Geodesics: Project $\gamma(t) \neq r(t)$



$$l_{\text{hyp}}(\gamma) = \int_0^1 |\gamma'(t)|_{\text{hyp}} dt = \int_0^1 \frac{|\gamma'(t)|}{\text{Im } \gamma(t)} dt$$

$$\geq \int_0^1 \frac{|r'(t)|_{\mathbb{H}}}{r(t)} dt = \int_1^K \frac{dt}{t}$$

$$= \log K$$

• $2\mathbb{H}$ is inf. for any.

Möbius send circles to circles on $\mathbb{C}$, so geodesics are met. lin or semicircle $\perp$ to $\mathbb{R}$.

Poincaré disk model:

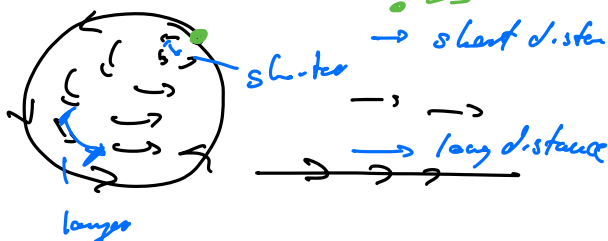
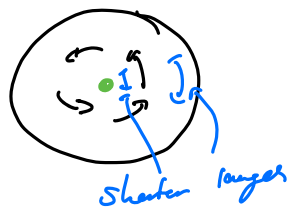


Hyperbolic:

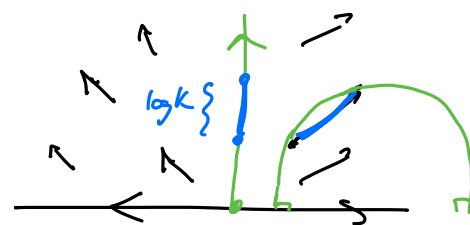
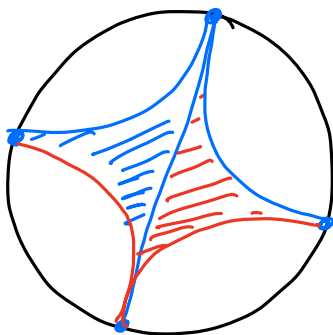
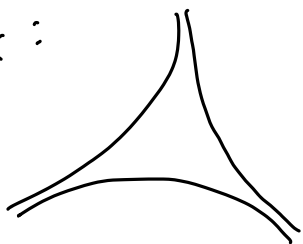


Elliptic:

Pseudocyclic

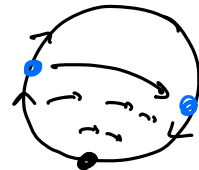


Ex:

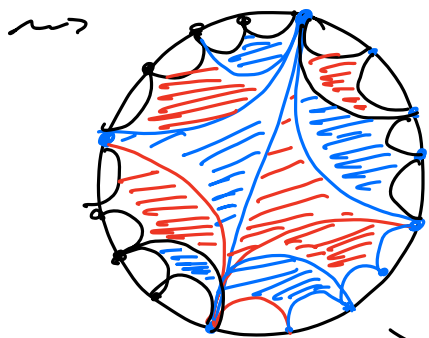


other points are not translated via geodesics

map real side via pseudocyclic: γ_1



same for blue: γ_2



Not actually equally spaced

$$\Delta(\infty, \infty, \infty) = \pi(2)$$

$$\Gamma = \langle \gamma_1, \gamma_2 \rangle \subset \text{PSL}(2, \mathbb{R})$$

$$\mathbb{D}/\Gamma \cong$$



Thm. (Pick) $f: X \rightarrow Y$ is mob, $X \neq Y$ are hyp. Riemann surfaces, then either:

1) $f: X \xrightarrow{\sim} Y$ is an isomorphism and strictly

2) f is a covering map but not 1-1, and

Ex: 1) $\text{Aut } H = \text{Isom}^+ H$

$$2) \begin{array}{ccc} w \in H & \xrightarrow{w \mapsto 2w} & H \\ \downarrow c \cdot w & & \downarrow \\ D - \{0\} & \xrightarrow{z \mapsto z^2} & D - \{0\} \end{array} \quad \begin{array}{l} \text{hyp.} \\ \text{isometry} \end{array}$$

coming up

$$3) \begin{array}{ccc} H & \longrightarrow & H \\ z & \longmapsto & z + i \end{array} \quad \text{constant distances}$$

Proof. $\begin{array}{ccc} H & \xrightarrow[\text{local isometries}]{\tilde{f}} & H \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} \tilde{x} & \longmapsto & \tilde{y} \\ \downarrow & & \downarrow \\ x & \longmapsto & y = f(x) \end{array} \quad |D\tilde{f}_{\tilde{x}}|_{\text{hyp}} = |Df_x|_{\text{hyp}}$

Pick $A \in \text{Isom}^+ H$, $A \cdot \tilde{y} = \tilde{x}$. By Schwarz lemma, $|D\tilde{f}_{\tilde{x}}| = |D(A \cdot \tilde{f})_{\tilde{x}}| \leq 1$
 w/ equality iff $\tilde{f}$ is an isomorphism.

Ex:  $\forall \subsetneq u$ is strictly contractive:
 $\ell_v(y) > \ell_u(y).$

Normal Families

Def. X, Y R.S.'s, $\mathcal{F} = \{f_n: X \rightarrow Y\}$ change loc. unif.
 if for every $K \subset X$, $K' \subset Y$ compact, $f_n(K) \cap K' = \emptyset$
 for n sufficiently large.

Def. X, Y R.S.'s. A set $\mathcal{F}$ of hol. functions $X \rightarrow Y$
 is normal if every sequence $\{f_n\} \subset \mathcal{F}$
 has a subsequence which converges loc. unif. or
 diverges which changes locally unif.

has a subsequence $n_k \rightarrow \infty$ of f_{n_k}

* if Y is compact, then is stronger:

$\{f_n\}$ norm $\Leftrightarrow$ eq. seq. in loc. unif. subseq.

Thm. X, Y are top. Every family $\mathcal{F}$ of maps $X \rightarrow Y$ is normal.

Proof. Pick $x_0 \in X$, consider $\mathcal{F}$ to eq. sequence.

1) $\overline{\{f(x_0) \mid f \in \mathcal{F}\}}$ is not contained in any compact set:

$\exists f_n \in \mathcal{F}$ st. for any $K' \subset Y$ compact,
 $d(f_n(x_0), K') \rightarrow \infty$.

For any $K \subset X$ compact, $x_1 \in K$, $y_0 \in K'$,

$$\begin{aligned} d(f_n(x_0), y_0) &\leq d(f_n(x_0), f_n(x_1)) + d(f_n(x_1), y_0) \\ \text{(Pick)} \quad &\leq d(x_0, x_1) + d(f_n(x_1), y_0) \end{aligned}$$

It follows that $d(f_n(x_1), y_0) \rightarrow \infty$, so f_n diverge loc. unif.

2) $\overline{\{f(x_0) \mid f \in \mathcal{F}\}}$ compact

$\{x_i\} \subset X$ countable dense subset

• Pick subsequence $\{f_{n_k}\}$ st. $\{f_{n_k}(x_0)\}$ converges.

• Pick: $d(f_n(x_0), f_n(x_1)) \leq d(x_0, x_1)$, so $\{f_n(x_i)\}$ lies in a compact. Pick subsequ... so that $f_{n_i}(x_i) \rightarrow g(x_i)$ as $n \rightarrow \infty$ $\forall i$.

• For each i , $\varepsilon > 0$, pick N st. $\forall n, n > N$,
 $d(f_n(x_i), f_{n_i}(x_i)) < \varepsilon/3$.

For x st. $d(x, x_i) < \varepsilon/3$,

$$d(f_n(x), f_n(x_i)) \leq d(f_n(x), f_n(x_i)) + d(f_n(x_i), f_{n_i}(x_i)) + d(f_{n_i}(x_i), f_{n_i}(x))$$

(Pick) $\leq \varepsilon \Rightarrow$ loc. unif Cauchy, loc unif conv. to a limit function.

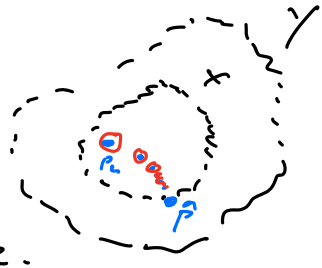
Lemma $X \subseteq Y$, Y hyp. $\{p_n\} \subset X$, $p_n \rightarrow \hat{p} \in \partial X \subset Y$. Then,

if $D_r(p_n) = \{x \mid d_X(x, p_n) < r\}$, $D_r(p_n) \rightarrow \hat{p}$ uniformly

p.f. $\tilde{X} \cong \mathbb{D} \hookrightarrow \mathbb{D}$, $f_n(\mathbb{D}) = D_r(p_n)$, $f_n(o) = p_n$

$\downarrow f_n$
 $X \leftarrow$

* Since $p_n \rightarrow \hat{p} \in \partial X$, for any compact $K \subset X$, $D_r(p_n) \cap K = \emptyset$ for n large.



• $K = \{a, b, c\}$: $Y \setminus \{a, b, c\}$ is hyp, so $\{f_n : \mathbb{D} \rightarrow Y \setminus \{a, b, c\}\}$ is a normal family $\Rightarrow f_{n_i} \rightarrow f$ loc. unif.

• $f(\mathbb{D}) = \hat{p}$ because otherwise $f(\mathbb{D}) \subset Y$ would be open and intersect X , but f_n leaves any compact set in X , and $X = \bigcup K_i$ of compact.

• Since we could have restricted the



subsequence from the subseq. loc. inf. conv. in $\mathbb{Z}$.

Lemma. $X, Y \subset \mathbb{Z}$ R.S. of $\{f_n: X \rightarrow Y\}$ converges
loc. inf in Y but not in $\mathbb{Z}$, then there is a subsequence
which conv. loc. inf to a pt of $\partial Y \subset \mathbb{Z}$.

Pf. $\{f_n\}$ does not converge in $\mathbb{Z}$, so $\exists K \subset X, K' \subset \mathbb{Z}$
compact so that $f_n(K) \cap K' \neq \emptyset$ for ∞ many n .

$\exists \{k_{n_i} \in K\} f_{n_i}(k_{n_i}) \rightarrow k' \in K'$.

1) If X, Y are hyp: $k_0 \in K, K \subset D_R(k_0)$, by Prop:
 $f_n(K) \subset D_R(f_n(k_0)) \rightarrow k' \in \partial Y$ inf by prop. Then.

2) Since f_n converge in Y , then is a compact $L \subset Y$ s.t.

$f_n(X) \subset Y \setminus L$ for n large and $Y \setminus L$ is hyp. Restrict

f_n to an open hyp $U \supset K$ and apply part (1) for
 $f_{n_i}: U \rightarrow Y \setminus L$.

Cor. $X, Y \subset \mathbb{Z}$ R.S.'s. Then a family of maps $X \rightarrow Y$
is normal iff it is normal as a family $X \rightarrow \mathbb{Z}$.

Pf. Normal $X \rightarrow Y$: $\begin{array}{l} \text{--- loc. inf conv in } Y \Rightarrow \text{loc. inf conv in } \mathbb{Z} \\ \text{--- loc. inf converge in } Y \end{array}$

$\Downarrow$
either loc. inf div. or conv in $\mathbb{Z}$

Normal $X \rightarrow \mathbb{Z}$:

loc. inf conv in $\mathbb{Z}$ & f : $\begin{array}{l} \text{--- } f(X) \subset Y: \text{loc. inf conv in } Y \\ \text{--- } f(X) \not\subset Y: \text{Pick hyp. set } U, \end{array}$

reason is before that $f_n \rightarrow \text{pt} \in \partial X$
 loc. unif. div. in Y

loc. unif. div. in $Z \Rightarrow$ loc. unif. div. in Y .

Cor. If $\mathcal{F}$ is a family of test maps $X \rightarrow \hat{\mathbb{C}}$ and there exists z_0 , then $\mathcal{F}$ is normal.

Def. $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ rational map. The Fatou set

$F(f)$ is the union of all open sets U where

$\{f^n|_U\}_{n \geq 0}$ is a normal family. The Julia set

$$J(f) := F(f)^c.$$

$\hookrightarrow z_0 \in \hat{\mathbb{C}}$; Either $\exists U \ni z_0$ st. $\{f^n|_U\}$ is normal, or

any seq. f^{n_j} has subseq. $\{f^{n_j}|_U\}$ conv. to unif.,
 or $\forall U \ni z_0, \exists$ subseq. $\{f^{n_j}|_U\}$ and $U \subset V$ cannot st.
 $\{f^{n_j}|_U\}$ does not conv. unif.

$\hookrightarrow F(f)$ is open, $J(f)$ is closed